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LETTER TO THE EDITOR

Can we always distinguish between positive and negative hierarchies?

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Abstract
It is a common belief that nonlinearizable PDEs in (1 + 1) dimensions cannot possess two mutually inverse *positive-order* recursion operators and that the negative hierarchies for such PDEs, unlike the positive ones, contain at most a *finite* number of local symmetries. We show that the equation $u_{xy} = uu_{xx} + \frac{1}{2}u_x^2 + u$, a generalization of the Hunter–Saxton equation considered by Manna and Neveu, provides a counterexample for both of these assertions. Namely, we find two positive-order integro-differential recursion operators for this equation and show that the corresponding positive *and* negative hierarchies consist solely of *local* symmetries. The recursion operators in question turn out to be mutually inverse on symmetries of the equation under study.

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1. Introduction

Consider the partial differential equation

$$u_{xy} = uu_{xx} + \tfrac{1}{2}u_x^2 + u. \qquad (1)$$

The transformation $u \mapsto -u$, $y \mapsto -y$ takes equation (1) into a particular case of the Manna–Neveu generalization [6] of the Hunter–Saxton equation [4, 5]. Manna and Neveu have already demonstrated integrability of this equation and, in particular, revealed its unusual relationship to the sinh-Gordon equation.

Oddly enough, the corresponding positive and negative hierarchies happen to consist of local symmetries alone. This property finds a natural explanation in the existence of two positive-order recursion operators being mutually inverse when restricted onto the hierarchies in question, and our first goal is to construct these operators. First of all, in sections 2 and 3 we reveal and investigate the relationship of (1) and the evolution equation

$$u_t = \frac{u_{xxx}}{(2u_{xx}+1)^{3/2}},$$

whose right-hand side happens to be a symmetry for (1). This evolution equation can be transformed into the Harry Dym equation, and this yields the first recursion operator for both (1) and the evolution equation in question.

The second recursion operator, presented in section 4, originates from the zero curvature representation for (1) with the matrices (4) through the procedure suggested in [7]. The relationship of these two operators is uncovered in section 5.

In spite of having positive order, each of our recursion operators decreases the order of an infinite set of symmetries. This is a highly unusual phenomenon that typically does not occur for the evolution equations, at least in $(1+1)$ dimensions, as readily follows from the theory of formal symmetries, cf e.g. [8, 11, 9]. This theory also shows that an evolution system cannot have two mutually inverse positive-order recursion operators.

Although the existence of two positive-order recursion operators being mutually inverse on symmetries is not quite surprising for hyperbolic PDEs (consider, e.g., a linear equation $u_{xy} = u$ and its recursion operators D_x and D_y), the hierarchy of *local* symmetries stretching beyond the zero symmetry in two directions seems to be rather unusual, even for the hyperbolic equations, and thus provides an interesting insight into how the integrable hierarchies can behave. Further research is required in order to understand when such a nonstandard behaviour can occur.

2. Symmetries

It is easy to show by direct computation that

$$\begin{aligned}
U_4 = {} & u^5 u_{xxxxx} - u_{yyyyy} + (10u^4u_x + 10u^3u_y)u_{xxxx} \\
& + \left(20u^4u_{xx} + 10u^2u_{yy} + \frac{55}{2}u^3u_x^2 + 50u^2u_xu_y + 15uu_y^2 + 10u^4\right)u_{xxx} \\
& + \left(5uu_{xx} + \frac{5}{2}u_x^2 + 5u\right)u_{yyy} + (40u^3u_x + 35u^2u_y)u_{xx}^2 \\
& + (25uu_x + 10u_y)u_{yy}u_{xx} \\
& + \left(20u^2u_x^3 + 50uu_x^2u_y + \left(20u_y^2 + 40u^3\right)u_x + 35u^2u_y\right)u_{xx} \\
& + \left(\frac{5}{2}u_x^3 + 15uu_x + 10u_y\right)u_{yy} + \frac{5}{4}u_x^4u_y + 10u^2u_x^3 + 25uu_x^2u_y \\
& + \left(\frac{25}{2}u_y^2 + 10u^3 - 1\right)u_x + \left(\frac{15}{2}u^2 - \frac{1}{2}\right)u_y,
\end{aligned}$$

$$U_3 = u^3u_{xxx} - u_{yyy} + (3u^2u_x + 3uu_y)u_{xx} + \frac{3}{2}u_x^2u_y + \left(\frac{3}{2}u^2 - \frac{1}{2}\right)u_x + 3uu_y,$$

$$U_2 = u_y,$$

$$U_1 = u_x,$$

$$U_{-1} = \frac{u_{xxx}}{(2u_{xx}+1)^{3/2}},$$

$$U_{-2} = \frac{u_{xxxxx}}{(2u_{xx}+1)^{5/2}} - 10\frac{u_{xxxx}u_{xxx}}{(2u_{xx}+1)^{7/2}} + \frac{35}{2}\frac{u_{xxx}^3}{(2u_{xx}+1)^{9/2}},$$

$$U_{-3} = \frac{u_{xxxxxxx}}{(2u_{xx}+1)^{7/2}} - 21\frac{u_{xxxxxx}u_{xxx}}{(2u_{xx}+1)^{9/2}} - \frac{7}{2}\frac{\left(101u_{xxxx} + 20u_{xx}u_{xxxx} - 69u_{xxx}^2\right)u_{xxxxx}}{(2u_{xx}+1)^{11/2}} + \frac{651}{2}\frac{u_{xxx}u_{xxxx}^2}{(2u_{xx}+1)^{11/2}} - 1848\frac{u_{xxx}^3u_{xxxx}}{(2u_{xx}+1)^{13/2}} + \frac{15\,015}{8}\frac{u_{xxx}^5}{(2u_{xx}+1)^{15/2}},$$

$$V = xu_x - yu_y - 2u$$

are symmetries of (1), that is they satisfy the linearization of (1), see e.g. [1] for more details. In particular, V is a scaling symmetry.

Upon setting $w = 2u + \frac{1}{2}x^2$ the symmetry U_{-1} becomes $\frac{1}{2}w_{xxx}w_{xx}^{-3/2}$, the right-hand side of the integrable evolution equation $w_t = w_{xxx}w_{xx}^{-3/2}$ which is known to be integrable, see e.g. [3]. Indeed, setting $r = 1/\sqrt{w}$ yields nothing but the well-known Harry Dym equation $r_t = r^3r_{xxx}$.

Conversely, consider the evolution equation

$$u_t = U_{-1} = \frac{u_{xxx}}{(2u_{xx}+1)^{3/2}} \tag{2}$$

and define a nonlocal variable ω by means of the formulae

$$\omega_t = -\frac{u_xu_{xxx} - 2u_{xx}^2 - u_{xx}}{(2u_{xx}+1)^{3/2}}, \qquad \omega_x = u - \frac{1}{2}u_x^2.$$

Then it is straightforward to verify that $G = uu_x + \omega$ is a nonlocal symmetry (or, in other terminology, a *shadow* of symmetry, cf e.g. [1] and references therein) for (2). Moreover, differentiating the equation $u_y = G$ with respect to x yields (1). Thus, not only the right-hand side of (2), U_{-1}, is a symmetry for (1), but (1) is, in a sense, a nonlocal symmetry for the evolution equation (2). This relationship of (1) and (2) is reminiscent of that of the sine-Gordon and the potential modified KdV equation, cf [2].

3. The first recursion operator

One can readily check that

$$\mathfrak{R} = \frac{1}{2u_{xx}+1}D_x^2 - \frac{u_{xxx}}{(2u_{xx}+1)^2}D_x - \frac{u_{xxx}^2}{(2u_{xx}+1)^3} + \frac{u_{xxx}}{(2u_{xx}+1)^{3/2}}D_x^{-1}\circ\frac{-3u_{xxx}^2 + 2u_{xxxx}u_{xx} + u_{xxxx}}{(2u_{xx}+1)^{5/2}} \tag{3}$$

is a common recursion operator for both equations (2) and (1).

More precisely, if W is a symmetry of (1), then so is

$$\mathfrak{R}\,W = \frac{W_{xx}}{2u_{xx}+1} - \frac{u_{xxx}W_x}{(2u_{xx}+1)^2} - \frac{u_{xxx}^2W}{(2u_{xx}+1)^3} - \frac{u_{xxx}Q}{(2u_{xx}+1)^{3/2}},$$

where the nonlocal variable Q is defined by means of the formulae

$$Q_x = \frac{3u_{xxx}^2 - 2u_{xxxx}u_{xx} - u_{xxxx}}{(2u_{xx}+1)^{5/2}}W,$$

$$Q_y = \frac{uu_{xxx}W_x}{(2u_{xx}+1)^{3/2}} - \frac{u_{xxx}W_y}{(2u_{xx}+1)^{3/2}} + \frac{\left(3u_{xxx}^2 - 2u_{xxxx}u_{xx} - u_{xxxx}\right)uW}{(2u_{xx}+1)^{5/2}}.$$

Likewise, if W is a symmetry of (2), then so is

$$\mathfrak{N}\,W = \frac{W_{xx}}{2u_{xx}+1} - \frac{u_{xxx}W_x}{(2u_{xx}+1)^2} - \frac{u_{xxx}^2 W}{(2u_{xx}+1)^3} - \frac{u_{xxx}\tilde{Q}}{(2u_{xx}+1)^{3/2}},$$

where the nonlocal variable $\tilde{Q}$ is defined by the formulae

$$\tilde{Q}_x = \frac{3u_{xxx}^2 - 2u_{xxxx}u_{xx} - u_{xxxx}}{(2u_{xx}+1)^{5/2}}W,$$

$$\begin{aligned}\tilde{Q}_t = {} & -\frac{W}{(2u_{xx}+1)^6}\big(8u_{xx}^3u_{xxxxxx} + 12u_{xx}^2u_{xxxxxx} + 6u_{xx}u_{xxxxxx} \\ & + u_{xxxxxx} - 60u_{xx}^2u_{xxx}u_{xxxxx} - 60u_{xx}u_{xxx}u_{xxxxx} - 15u_{xxx}u_{xxxxx} - 9u_{xxxx}^2 \\ & - 36u_{xx}^2u_{xxxx}^2 - 36u_{xx}u_{xxxx}^2 + 234u_{xx}u_{xxx}^2u_{xxxx} + 117u_{xxx}^2u_{xxxx} - 150u_{xxx}^4\big) \\ & + \frac{W_x}{(2u_{xx}+1)^5}\big(4u_{xx}^2u_{xxxxx} + 4u_{xx}u_{xxxxx} + u_{xxxxx} - 18u_{xx}u_{xxx}u_{xxxx} \\ & - 9u_{xxx}u_{xxxx} + 15u_{xxx}^3\big) + \frac{\left(-2u_{xx}u_{xxxx} - u_{xxxx} + 3u_{xxx}^2\right)W_{xx}}{(2u_{xx}+1)^4}.\end{aligned}$$

So, we can enlarge the list of symmetries U_{-j} downward by setting $U_{-j} = \mathfrak{N}^{j-1}(U_{-1})$ for all $j = 2, 3, \ldots$.

The operator $\mathfrak{N}$ is hereditary because it can be obtained from the well-known hereditary recursion operator of the Harry Dym equation $r_t = r^3 r_{xxx}$ via the transformation $r = 1/\sqrt{2u_{xx}+1}$. It can be further shown that $\mathfrak{N}$ and the symmetry $U_{-1} = u_{xxx}/(2u_{xx}+1)^{3/2}$ of (1) meet the requirements of theorem 1 of [10], and hence the symmetries U_{-j}, $j \in \mathbb{N}$, of (1) are local.

4. The second recursion operator

Equation (1) has a zero curvature representation (ZCR) with the matrices

$$A = \begin{pmatrix} \frac{1}{2}\lambda u_x & 1 \\ \frac{1}{4}\lambda\left(1-\lambda u_x^2\right) & -\frac{1}{2}\lambda u_x \end{pmatrix}, \qquad B = \begin{pmatrix} \frac{1}{2}u_x u\lambda & \frac{\lambda u+1}{\lambda} \\ \frac{1}{4}\left(1-\lambda u-\lambda^2 u u_x^2\right) & -\frac{1}{2}u_x u\lambda \end{pmatrix} \tag{4}$$

that satisfy $A_y - B_x + [A, B] = 0$ as a consequence of equation (1); λ is a spectral parameter.

One more recursion operator for (1) can be obtained from the ZCR by the method suggested in [7]. For the sake of brevity, we present here only the final result of corresponding computations.

Let us have a symmetry W of (1). Put

$$\mathfrak{R}\,W = u_x P - uW + P_y,$$

where P solves the system

$$P_x = W,$$

$$P_y = -2uu_{xx}W - u_xW_y - u_yW_x - u^2W_{xx} - uW + W_{yy}.$$

It is easy to show that $\mathfrak{R}\,W$ is again a symmetry. Hence, the mapping $\mathfrak{R}$ is a recursion operator of (1). Note that $\mathfrak{R}\,W$ is defined uniquely up to the addition of a constant multiple of $U_1 = u_x$.

Enlarge the list of symmetries U_i upward by putting $U_i = \mathfrak{R}^{i-1}(U_1)$ for all $i = 2, 3, \ldots$. To prove locality of the symmetries U_i we shall employ theorem 1 of [10]. First, note that (1) can be rewritten in the evolutionary form as

$$u_x = v, \qquad v_x = \frac{v_y - \frac{1}{2}v^2 - u}{u}. \tag{5}$$

The operator $\mathfrak{R}$ yields, in an obvious way, a recursion operator $\tilde{\mathfrak{R}}$ for (5) which can be written in the standard pseudo-differential form as

$$\tilde{\mathfrak{R}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} D_y^2 + \begin{pmatrix} 0 & -u \\ \frac{2v_y - v^2}{2u} & 0 \end{pmatrix} D_y + \begin{pmatrix} -v_y - \frac{1}{2}v^2 & -u_y \\ v\frac{-2v_y + v^2 + 2u}{2u} & -v_y + \frac{1}{2}v^2 \end{pmatrix} + \begin{pmatrix} v/2 \\ \frac{v_y - \frac{1}{2}v^2 - u}{2u} \end{pmatrix} D_y^{-1} \circ (v^2 + 2u, 2vu). \tag{6}$$

A straightforward but tedious computation shows that the operator $\tilde{\mathfrak{R}}$ is hereditary, and $\tilde{\mathfrak{R}}$ along with the symmetry

$$K = \begin{pmatrix} u_y \\ v_y \end{pmatrix}$$

of (5) meet the requirements of theorem 1 of [10], and hence the symmetries $K_j = \tilde{\mathfrak{R}}^j(K)$ are local for all $j = 1, 2, \ldots$. It is immediate that upon going back from (5) to (1) the symmetries K_i go into U_{i+2}, and hence U_i are local for all $i = 2, 3, \ldots$.

5. Unification

Proposition 1. *The recursion operators $\mathfrak{R}$ and $\mathfrak{N}$ are mutually inverse modulo linear span of U_1 and U_{-1}.*

More precisely, for any symmetry W of (1) we have

$$\mathfrak{R}\,\mathfrak{N}(W) = W + c_1 U_1, \qquad \mathfrak{N}\,\mathfrak{R}(W) = W + c_{-1} U_{-1},$$

where c_1, c_{-1} are arbitrary integration constants.

If we agree to set all integration constants to zero, we have simply $\mathfrak{R}\,\mathfrak{N}(W) = W$ and $\mathfrak{N}\,\mathfrak{R}(W) = W$. With this convention in mind, we have, upon setting $U_0 = 0$,

$$\mathfrak{R}(U_i) = U_{i+1}, \qquad \mathfrak{N}(U_i) = U_{i-1}$$

for all $i \in \mathbb{Z}$. It is a remarkable fact that the order of symmetries with negative (resp. positive) indexes is lowered by applying $\mathfrak{R}$ (resp. $\mathfrak{N}$) and, conversely, $\mathfrak{R}$ (resp. $\mathfrak{N}$) increases the order for the symmetries with positive (resp. negative) indices. This is readily seen from expressions (3) and (6) for $\mathfrak{N}$ and $\tilde{\mathfrak{R}}$ in pseudo-differential form.

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